Abstract Algebra I
Exercise Sheet 3: Bijections and Homomorphisms

Bijections

Recall that a function $f : X \to Y$ is *injective* if every two elements of the domain X are mapped to distinct elements of the range.

Example.

- $f : \mathbb{R} \to \mathbb{R} : x \mapsto 0$ is not injective as $f(1) = 0$ and $f(2) = 0$.
- $f : \mathbb{R} \to \mathbb{R} : x \mapsto x$ is injective as $f(x) = f(y)$ if and only if $x = y$, i.e. two numbers x and y are mapped to distinct numbers if and only if x and y are themselves distinct.

Recall that a function $f : X \to Y$ is *surjective* if every element of the range Y is mapped onto by some value of the domain.

Example.

- $f : \mathbb{R} \to \mathbb{R} : x \mapsto 0$ is not injective as no element maps to 1, i.e. $f(x) \neq 1$ for any $x \in \mathbb{R}$.
- $f : \mathbb{R} \to \mathbb{R} : x \mapsto 2x$ is surjective as for every element y in $\mathbb{R}$ there is a value that is mapped to y by f, in particular $f(y/2) = y$.

One can check for surjectivity of a function by computing its image.

Theorem. *A function $f : X \to Y$ is surjective if and only if* $\mathrm{Im}(f) = Y$.

Question 1. Decide whether the following functions are injective, surjective, and bijective. If yes, give a proof, if no, give a counterexample.

- $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$
- $f : \mathbb{R} \to \mathbb{R}_{\geq 0} : x \mapsto x^2$
- $f : \mathbb{R} \to \mathbb{R} : x \mapsto e^x$
- $f : \mathbb{R} \to \mathbb{R}_{\geq 0} : x \mapsto e^x$
- $f : \mathbb{R} \to \mathbb{R}_{>0} : x \mapsto e^x$
- $f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto 2x$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto x + 1$
- $f : \mathbb{Z} \to 2\mathbb{Z} : x \mapsto 2x$
- $f : \mathbb{Z} \to \mathbb{N} : x \mapsto |x|$
- $f : \mathbb{N} \to \mathbb{N} : x \mapsto x + 1$
- $f : \mathbb{N} \to \mathbb{Z} : x \mapsto x$

Recall a homomorphism is a function between groups where the group operations may be computed before or after the function is applied, i.e. $f(ab) = f(a)f(b)$. Recall that $\mathbb{R}^\times$ is the group of units in $\mathbb{R}$, that is, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

One can check for injectivity of a homomorphism by computing its kernel.

Theorem. *A homomorphism $f : X \to Y$ is injective if and only if* $\ker(f) = \{e_X\}$.

Question 2. Decide whether each of the following functions are homomorphisms. If yes, give a proof and determine what type of homomorphism they are (homomorphism, isomorphism, automorphism, etc.). If no, provide a counterexample. The group operation is assumed to be addition unless otherwise specified.

- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto 0$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto 1$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto x$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto 2x$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto x + 1$
- $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto -x$
- $f : \mathbb{Z} \to 2\mathbb{Z} : x \mapsto 2x$
- $f : \mathbb{Z} \to 2\mathbb{Z} : x \mapsto 4x$
- $f : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot) : x \mapsto e^x$
- Let $\mathcal{M}_n(k)$ denote $n \times n$ matrices with entries in k. Let M^+ denote the transpose of M.
$$f : \big(\{M \in \mathcal{M}_n(\mathbb{R}) \,|\, \det(M) \neq 0\}, \cdot\big) \to \big(\{M \in \mathcal{M}_n(\mathbb{R}) \,|\, \det(M) \neq 0\}, \cdot\big) : M \mapsto M^+.$$
- Assuming usual properties of determinants:
$$\det : \big(\{M \in \mathcal{M}_n(\mathbb{R}) \,|\, \det(M) \neq 0\}, \cdot\big) \to \mathbb{R}^\times$$
- Assuming usual properties of traces:
$$\mathrm{trace} : \big(\{M \in \mathcal{M}_n(\mathbb{R}) \,|\, \det(M) \neq 0\}, \cdot\big) \to (\mathbb{R}, +)$$
- $$\begin{aligned} f :& \big(\{M \in \mathcal{M}_n(\mathbb{Q}) \,|\, (M_{ij}) = \frac{z_{ij}}{n},\ z_{ij} \in \mathbb{Z}\ \det(M) = 1\}, \cdot\big) \to \big(\{M \in \mathcal{M}_n(\mathbb{Z})\}, +\big) \\ :& (M)_{ij} \mapsto z_{ij} \end{aligned}$$

Question 3. Let $\varphi : G_1 \to G_2$ be an isomorphism. Prove that the inverse map $\varphi^{-1} : G_2 \to G_1$ is also an isomorphism.

Question 4. Find two distinct automorphisms of $(\mathbb{Z}, +)$.

Question 5. Recall that $(\mathbb{Z}_n, +)$ is the group made up of $\{0, 1, \ldots, n-1\}$ with addition defined modulo n, and that $(\mathbb{Z}_n, \cdot)$ is the group made up of $\{1, \ldots, n-1, n\}$ with multiplication defined modulo n. Decide whether the two groups $(\mathbb{Z}_6, +)$ and $(\mathbb{Z}_6, \cdot)$ are isomorphic. Decide for which n the groups $(\mathbb{Z}_n, +)$ and $(\mathbb{Z}_n, \cdot)$ are isomorphic.

Question 6. Decide whether $(\mathbb{R}, +)$ and $(\mathbb{R}^\times, \cdot)$ are isomorphic.